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BOOTSTRAPPING THE SAMPLE MEAN FOR DATA WITH INFINITE VARIANCE

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Technical Report No. 296

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Abstract: When data comes from a distribution belonging to the domain of attraction of a stable law, Athreya (1987a) showed that the bootstrapped sample mean has a random limiting distribution, implying that the naive bootstrap could fail in the heavy-tailed case. The goal here is to classify all possible limiting distributions of the bootstrapped sample mean when the sample comes from a distribution with infinite variance, allowing the broadest possible setting for the (nonrandom) scaling, the resample size, and the mode of convergence (in law). The limiting distributions turn out to be infinitely divisible with possibly random Lévy measure, depending on the resample size. An averaged-bootstrap algorithm is then introduced which eliminates any randomness in the limiting distribution. Finally, it is shown that (on the average) the limiting distribution of the bootstrapped sample mean is stable if and only if the sample is taken from a distribution in the domain of (partial) attraction of a stable law.

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1. INTRODUCTION

The bootstrap (Efron (1979)) is a resampling technique for nonparametrically modeling the distribution of a general statistic $T_n = T_n(\mathcal{X}_n;F)$, where $\mathcal{X}_n = (X_1,\ldots,X_n)$ is a random sample of size n from a population distribution F. Bootstrap "replicates" $T_n^* = T_{m_n}(\mathcal{Y}_{m_n};F_n)$ of the statistic are generated by resampling $\mathcal{Y}_{m_n} = (Y_1,\ldots,Y_{m_n})$ from the empirical distribution F_n of $\{X_1,\ldots,X_n\}$, i.e., the Y_i's are conditionally iid with distribution having mass $\frac{1}{n}$ at each of the points $X_1,\ldots,X_n$. The conditional distribution $\mathcal{L}(T_n^*|\mathcal{X}_n)$, which is necessarily random, is intended to approximate the true nonrandom distribution $\mathcal{L}(T_n)$.

In the present paper we deal only with the simplest statistic T_n, the standardized sample mean. The sample mean is the easiest statistic to handle analytically, and if the bootstrap algorithm fails for this simple statistic, we cannot expect it to succeed for more complex statistics. Our work is motivated by the results of Bickel and Freedman (1981), Singh (1981), Athreya (1986a,b, 1987a,b), Giné and Zinn (1990a,b), Arcones and Giné (1989), Hall (1990), and Knight (1989).

We denote by $H_n(x)$ the conditional distribution function of T_n^* given $\mathcal{X}_n$, $H_n(x) = P(T_n^* \leq x \mid \mathcal{X}_n)$, where $T_n^* = a_n^{-1} \Sigma_{i=1}^{m_n}(Y_i-c_n)$ is the bootstrap version of the sample mean with possibly data-driven scaling a_n and centering c_n ($c_n = \overline{X}_n$ when F has finite mean, and otherwise $c_n = 0$), and with (nonrandom) resample size m_n (which may differ from the original sample size n). As $H_n(x)$ is a random distribution function, several modes of convergence are possible (such as convergence a.s., in probability, in law), and the limit may be random or nonrandom. We judge the performance of the bootstrap by asymptotically comparing H_n to $\mathcal{L}_\infty$, the limiting distribution of the standardized sample mean T_n. We now review some of the known results on the bootstrap. Each of these results involves specific assumptions on a_n and c_n, for which the reader is referred to the original papers.

A. (<u>Sufficient conditions</u>.)

(A.1) (Bickel and Freedman (1981)) If F has finite variance and $m_n \longrightarrow \infty$, then

$$\sup_{x\in\mathbb{R}} |H_n(x) - \mathcal{L}_\infty(x)| \xrightarrow[n\to\infty]{a.s.} 0;$$

here $\mathcal{L}_\infty$ is the standard normal distribution.

(A.2) (Arcones and Giné (1989)) If F belongs to the domain of attraction of the normal law and $m_n \longrightarrow \infty$, then

$$\sup_{x\in\mathbb{R}} |H_n(x) - \mathcal{L}_\infty(x)| \xrightarrow[n\to\infty]{P} 0;$$

here $\mathcal{L}_\infty$ is the standard normal distribution. (See Athreya (1987b) for $m_n=n$.)

(A.3) If F belongs to the domain of attraction of an α-stable law, $0<\alpha<2$, then

(i) (Athreya (1987a)) if $m_n=n$, the bootstrap fails (in fact $H_n(x)$ converges in law to a random distribution which differs from $\mathcal{L}_\infty$, an α-stable distribution);

(ii) (Athreya (1987b)) if $m_n/n \longrightarrow 0$, the bootstrap works in probability, i.e., the conclusion of (A.2) holds with an α-stable law $\mathcal{L}_\infty$;

(iii) (Arcones and Giné (1989)) if (m_n/n) loglog $n \longrightarrow 0$, the bootstrap works a.s., i.e., the conclusion of (A.1) holds with an α-stable law $\mathcal{L}_\infty$.

(A.4) (Hall (1990)) If F has slowly varying tails and one tail completely dominates the other, and $m_n=n$, then the bootstrap fails (in fact H_n converges in probability to a nonrandom Poisson distribution).

B. (Necessary conditions.)

(B.1) (Arcones and Giné (1989)) If $\inf m_n/n > 0$ and for some possibly random distribution $H(x)$,

$$H_n(\cdot) \xrightarrow[n\to\infty]{w} H(\cdot) \quad \text{a.s.}$$

then F has finite variance, and H is $\mathcal{L}_\infty$ (the standard normal distribution). (See Giné and Zinn (1990b) for $m_n=n$).

(B.2) (Arcones and Giné (1989)) If $m_n \longrightarrow \infty$, a_n are nonrandom, and for some possibly random distribution $H(x)$,

$$H_n(\cdot) \xrightarrow[n\to\infty]{w} H(\cdot) \quad \text{in probability,}$$

then F belongs to some domain of partial attraction and H is $\mathcal{L}_\infty$ (an infinitely divisible distribution). If furthermore $\liminf_{n\to\infty} m_n/n > 0$, then F belongs to the domain of attraction of a normal law, and H is $\mathcal{L}_\infty$ (the standard normal distribution).

(B.3) (Hall (1990)) If $m_n=n$, a_n are possibly random, and for some nonrandom distribution $H(x)$,

$$H_n(\cdot) \xrightarrow[n\to\infty]{} H(\cdot) \quad \text{in probability (at continuity points of H),}$$

then either F belongs to the domain of attraction of a normal law and H is $\mathcal{L}_\infty$ (the standard normal distribution), or F has slowly varying tails with one tail completely dominating the other and H is a Poisson distribution.

Our interest is to extract information from H_n in the broadest possible setting, i.e., with minimal restrictions on the underlying distribution F, the resample size m_n, and the standardization a_n and c_n, and allowing the weakest form of convergence (in law) to a possibly random limit. Our goal is to classify all possible limiting distributions, and then to eliminate any randomness in the limit.

When F is general with infinite variance, assuming nonrandom a_n, we give in Section 2 necessary and sufficient conditions for weak convergence of the finite dimensional distributions of $\{H_n(x), -\infty<x<\infty\}$, and we classify all possible limits. The result (Theorem 1) shows that H_n may have a limit even when the sample mean does not, and that the limit of H_n is always infinitely divisible with possibly random Lévy measure, depending on the choice of the resample size m_n.

For the case where $H_n(x)$ has a random limit, we introduce in Section 3 an averaged-bootstrap algorithm and show in Theorem 2 that the averaged-bootstrap distribution converges a.s. to a nonrandom limit, G(x), which is the expected value of the random limit of $H_n(x)$. If G contains useful information about $\mathcal{L}_\infty$, then averaged-bootstrap resampling is useful, i.e., bootstrap resampling is still worthwhile on the average.

In Section 4 we apply these results to population distributions in the domain of attraction of a stable law. We show that the random limiting distribution of $H_n(x)$ obtained by Athreya (1987a) with a data-based scaling, is not stable even on the average (Theorem 4). However, with a nonrandom scaling, the averaged bootstrap actually yields a stable distribution with the same index as $\mathcal{L}_\infty$ but with possibly different skewness and scale parameters (Theorem 5). Appropriately adjusted, the averaged-bootstrap algorithm works a.s. in some cases, i.e., yields the correct skewness and scale of $\mathcal{L}_\infty$ (Theorems 6 and 7). Moreover the limit distribution G of the averaged bootstrap is stable if and only if F belongs to the domain of (partial) attraction of a stable law (Theorem 8).

2. GENERAL POPULATION DISTRIBUTION WITH INFINITE VARIANCE

A fundamental question is: What are the possible limits of the bootstrapped conditional distribution $\mathcal{L}(T_n^*|\mathcal{X}_n)$ when we do not have any restriction on the population distribution F, the resample size m_n, and the scaling a_n. We consider on the one hand general population distributions F with infinite variance (i.e., without the domain of attraction restriction of results A), and on the other hand we allow general resample size m_n and convergence in distribution of H_n (rather than the more restrictive a.s. or in probability convergence of results B).

The following theorem gives necessary and sufficient conditions, explicitly in terms of the univariate distribution F of X, for the convergence in law of $\mathcal{L}(T_n^*|\mathcal{X}_n)$ and describes the possible limits.

Theorem 1. Assume the population distribution F has infinite variance, a_n is nonrandom,

$$(1.0) \qquad m_n \longrightarrow \infty, \quad a_n \longrightarrow \infty, \quad m_n a_n^{-2} \longrightarrow 0 \qquad \text{as } n \longrightarrow \infty,$$

and when F has finite mean, assume furthermore

$$(1.1) \qquad \lim_{M\to\infty} \lim_{n\to\infty} n\, P\{|X| > a_n \max(M, \epsilon n/m_n)\} = 0 \quad \text{for all } \epsilon > 0,$$

$$(1.2) \qquad \lim_{M\to\infty} \lim_{n\to\infty} m_n a_n^{-1} E\{|X| I(|X| \geq M a_n,\ |X| \leq \epsilon a_n n/m_n)\} = 0 \quad \text{for some } \epsilon > 0.$$

Then

$$(1.3) \qquad \{H_n(x_1), \ldots, H_n(x_k)\} \xrightarrow[n\to\infty]{\mathcal{L}} \{H(x_1), \ldots, H(x_k)\}$$

for some possibly random distribution function H(x), any integer k and $x_1, \ldots, x_k \in \mathbb{R}$, if and only if the following three conditions (1.a)-(1.c) are satisfied:

(1.a) There exists a Lévy measure λ, such that for all $y>0$,

$$\lim_{n\to\infty} m_n P(a_n^{-1}X > y) = \lambda[y,\infty)$$
$$\lim_{n\to\infty} m_n P(a_n^{-1}X < -y) = \lambda(-\infty,-y] \qquad \text{if } m_n/n \xrightarrow[n\to\infty]{} c \in [0,\infty),$$

$$\lim_{n\to\infty} n\, P(a_n^{-1}|X| > y) = 0 \qquad \text{if } \varliminf_{n\to\infty} m_n/n > 0,\ m_n/n \not\to c \in [0,\infty),$$

or

$$\lim_{n\to\infty} m_n P(a_n^{-1}|X| > y) = 0 \qquad \text{if } 0 = \varliminf_{n\to\infty} m_n/n < \varlimsup_{n\to\infty} m_n/n.$$

(1.b) There exists a Lévy measure ν on $(0,\infty)$ such that

$$\lim_{\epsilon\to 0}\lim_{n\to\infty} n\, P(|X| \le a_n\epsilon,\ |X| > a_n\sqrt{\delta n/m_n}\ ) = \nu(\delta,\infty)$$

for every $\delta > 0$ such that $\nu\{\delta\} = 0$.

(1.c) For all $\delta > 0$ such that $\nu\{\delta\} = 0$, there exists σ_δ such that

$$\lim_{\epsilon\to 0}\lim_{n\to\infty} m_n a_n^{-2} E\{X^2 I[|X| \le a_n \min(\epsilon,\sqrt{\delta n/m_n})]\} = \sigma_\delta^2.$$

In this case, the possibly random characteristic function $\varphi_H(\cdot)$ corresponding to $H(\cdot)$ is

$$(1.4)\quad \varphi_H(t) = \begin{cases} \exp\{-\frac{1}{2}t^2\sigma^2 + \int_{-\infty}^{\infty}[e^{itx}-1-it\tau(x)]d\lambda(x)\ \} & \text{if } \frac{m_n}{n} \longrightarrow 0, \\ \exp\{-\frac{1}{2}t^2\sigma^2 + c\int_{-\infty}^{\infty}[e^{itx}-1-it\tau(x)]dN(x)\ \} & \text{if } \frac{m_n}{n} \longrightarrow c \in (0,\infty), \\ \exp\{-\frac{1}{2}t^2\sigma^2 - \frac{1}{2}t^2 W\ \} & \text{otherwise,} \end{cases}$$

where $\sigma^2 = \lim_{\delta\downarrow 0}\sigma_\delta^2$, $\tau(x) = x$ if F has finite mean, and $\tau(x) = 0$ otherwise, N is a Poisson random measure with intensity measure $c^{-1}\lambda$, $\sigma^2+W \ge 0$, and the characteristic function $\varphi_W(u)$ of W is $\exp\{\int_0^\infty(e^{iux}-1)d\nu(x)\}$. Furthermore, the expected value G(x) of H(x) has characteristic function $\varphi_G(t)$ $(= E\varphi_H(t))$ which is always infinitely divisible with the following form

$$(1.5)\qquad \varphi_G(t)=\begin{cases}\exp\{-\frac{1}{2}t^2\sigma^2+\int_{-\infty}^{\infty}[e^{itx}-1-it\tau(x)]d\lambda(x)\ \} & \text{if } \frac{m_n}{n}\longrightarrow 0,\\ \exp\{-\frac{1}{2}t^2\sigma^2+c^{-1}\int_{-\infty}^{\infty}\{\exp[c(e^{itx}-1-it\tau(x))]-1\}d\lambda(x)\} & \text{if } \frac{m_n}{n}\longrightarrow c\in(0,\infty)\\ \exp\{-\frac{1}{2}t^2\sigma^2\}\ E\exp\{-\frac{1}{2}t^2W\} & \text{otherwise.}\end{cases}$$

Theorem 1 classifies all possible weak limits of $\mathcal{L}(T_n^*|\mathfrak{X}_n)$. They are either the law of an independent sum of a Gaussian r.v. and a compound Poisson r.v. with possibly random Lévy measure, or else the law of an infinitely divisible scale mixture of a Gaussian r.v., i.e., a Gaussian law with zero mean and random infinitely divisible variance. The (possibly random) variance of the Gaussian component and the Lévy measure of the compound Poisson component depend on the choice of resample size and standardization.

Conditions (1.1) - (1.2) ensure that the tails of the integrals in (1.4) when F has finite mean are eventually negligible. The Gaussian component of the limit has variance σ^2 determined by the asymptotic behavior of a certain truncated second moment of F, described in (1.c). The compound Poisson component of the limit is determined by the tail behavior of F described in (1.a). And the Gaussian mixture has an infinitely divisible variance whose Lévy measure ν is determined by the asymptotic behavior of a certain truncated probability described in (1.b).

The form of the limits (1.4) and (1.5) can be interpreted as follows. The bootstrap resample comes from the empirical distribution F_n. If we take the resample size m_n much smaller than the original sample size n ($m_n/n \longrightarrow 0$), then from the perspective of the resample $\mathcal{Y}_{m_n}$ (m_n large), F_n is essentially like the nonrandom population distribution F. Thus the distribution of the bootstrapped sample mean will yield a deterministic limit if it exists (as in (1.4)). This phenomenon appears in the literature in other contexts (see Bretagnolle (1983) and Swanepoel (1986)).

Another interesting effect arises from the fact that F_n has finite support (trivially symmetric tails). If we take the resample size m_n much larger than the original sample size n ($m_n/n \longrightarrow \infty$), then $\mathcal{Y}_{m_n}$ is like a very large sample from F_n (rather than F). Hence the structure of F_n (symmetric tails) will be reflected in the limiting distribution H, which indeed is always symmetric in this case (provided it exists).

Under conditions (1.a)-(1.c), the limit H, or its expected value G, may contain some information about $\mathcal{L}_\infty$. This suggests that H_n, or its expected value EH_n, may contain some information about $\mathcal{L}(T_n)$. In Section 3, we develop the averaged-bootstrap algorithm in order to extract this information.

There are many distribution functions with infinite variance which do not belong to the domain of attraction of any stable law, i.e., whose sample mean does not have a limiting distribution. Theorem 1 suggests the existence of a limit for $\mathcal{L}(T_n^*|\mathcal{X}_n)$ in some such cases.

3. AVERAGED BOOTSTRAP

In Section 2 we have classified all possible limiting (conditional) distributions of T_n^*. It is clear from Theorem 1 that the limits may be random, and therefore different from the nonrandom limit $\mathcal{L}_\infty$ of $\mathcal{L}(T_n)$ when it exists, e.g., when F belongs to some stable domain of attraction. The question arises: How can we extract useful information about $\mathcal{L}_\infty$? For this purpose we now introduce the averaged-bootstrap algorithm.

Partition the data set into ℓ_n blocks, each having k_n observations, $n=\ell_n k_n$, as follows:

$$\mathcal{X}_n = (\underbrace{X_1,\ldots,X_{k_n}}_{{}_1\mathcal{X}_{k_n}},\underbrace{X_{k_n+1},\ldots,X_{2k_n}}_{{}_2\mathcal{X}_{k_n}},\ \ldots\ ,\underbrace{X_{(\ell_n-1)k_n+1},\ldots,X_{\ell_n k_n}}_{{}_{\ell_n}\mathcal{X}_{k_n}}).$$

Instead of bootstrapping the entire sample $\mathcal{X}_n$ to obtain $T_n^* = a_n^{-1}\Sigma_{i=1}^{m_n}(Y_i - c_n)$, we apply the bootstrap algorithm *within each block* $j=1,\ldots,\ell_n$ to obtain a block-bootstrapped version of the sample mean

$$ {}_jT^*_{k_n} = {}_ja^{-1}_{k_n} \sum_{i=1}^{m_{k_n}} ({}_jY_i - {}_jc_{k_n}), $$

where $\{{}_jY_i,\ 1 \le i \le m_{k_n}\}$ are resampled from the j^{th} block ${}_j\mathcal{X}_{k_n}$, i.e., they are conditionally independent with common distribution

$$ P({}_jY_1 = X_{(j-1)k_n+i} \mid {}_j\mathcal{X}_{k_n}) = \frac{1}{k_n}, \quad 1 \le i \le k_n. $$

The conditional distribution of the block-bootstrapped sample mean ${}_jT^*_{k_n}$ is denoted by

$$ {}_jH_{k_n}(x) = P({}_jT^*_{k_n} \le x \mid {}_j\mathcal{X}_{k_n}). $$

In T_n^* we use $c_n = \overline{X}_n$ if F has finite mean, $c_n = 0$ otherwise; and in ${}_jT^*_{k_n}$ we use ${}_jc_{k_n} = {}_j\overline{X}_{k_n}$ (the sample mean in the j^{th} block) if F has finite mean, ${}_jc_{k_n} = 0$ otherwise. For a data-driven scaling, we use ${}_ja_{k_n} = a_{k_n}({}_j\mathcal{X}_{k_n})$ in ${}_jT^*_{k_n}$, analogously to using $a_n = a_n(\mathcal{X}_n)$ in T_n^*.

Note that $\{{}_jH_{k_n}(x),\ 1 \le j \le \ell_n\}$ are iid random distributions; their average is denoted by

$$ \overline{H}_n(x) = \frac{1}{\ell_n} \sum_{j=1}^{\ell_n} {}_jH_{k_n}(x), $$

and is referred to as the averaged-bootstrap distribution.

We now show that the averaged-bootstrap algorithm converges a.s. when the standard bootstrap algorithm converges in law (the weakest possible mode).

Theorem 2. If $\ell_n \cong n^{\delta}$, $0<\delta<1$, and for each $x \in \mathbb{R}$,

$$H_n(x) \xrightarrow[n\to\infty]{\mathcal{L}} H(x),$$

where $H(x)$ is a possibly random distribution function, then for each $x \in \mathbb{R}$

$$\bar{H}_n(x) \xrightarrow[n\to\infty]{a.s.} G(x),$$

where $G(x) = EH(x)$. Furthermore if $G(x)$ is continuous, then the convergence is uniform in x.

Thus the averaged-bootstrap algorithm produces a nonrandom limiting distribution $G(x)$, and it is of interest to investigate the relationship between G and $\mathcal{L}_\infty$. If G and $\mathcal{L}_\infty$ agree, then the averaged bootstrap works a.s. and $\bar{H}_n$ contains useful information about $\mathcal{L}(T_n)$.

4. APPLICATIONS TO POPULATION DISTRIBUTIONS IN DOMAINS OF ATTRACTION

In this section we apply Theorems 1 and 2 to population distributions in the domain of attraction of a stable law. Each of the stated results (except Theorem 3) includes two parts. First we establish convergence of the random distribution $H_n(x)$ to a possibly random limiting distribution $H(x)$, using Theorem 1 when appropriate. Secondly, an explicit expression for $G(x) = EH(x)$ is obtained, again using Theorem 1. By comparing G with $\mathcal{L}_\infty$, we can in each case determine whether the averaged bootstrap ($\bar{H}_n$) is of any practical use (via Theorem 2).

Before stating our results we briefly review the stable laws $S(\alpha,\beta,s)$ with index $\alpha \in (0,2]$, skewness $\beta \in [-1,1]$, scale parameter $s \geq 0$ (the actual scale is $s^{1/\alpha}$; here for simplicity we use s instead), and characteristic function:

$$\varphi(t) = \begin{cases} \exp\{-s|t|^{\alpha}[1 - i\beta\tan(\alpha\pi/2)\mathrm{sgn}(t)]\}, & \alpha \neq 1, \\ \exp\{-s|t|[1 + i\beta(\frac{2}{\pi})\log|t|\mathrm{sgn}(t)]\}, & \alpha = 1. \end{cases}$$

A distribution function F belongs to the domain of attraction of a stable law with $0<\alpha<2$, denoted $F \in D(S(\alpha,\beta,s))$, if and only if there exist constants $c_1, c_2 \geq 0$, and a slowly varying function $L(x)$ such that $1-F(x) \sim c_1 x^{-\alpha} L(x)$, $F(-x) \sim c_2 x^{-\alpha} L(x)$, as $x \longrightarrow \infty$, where $\beta = (c_1-c_2)/(c_1+c_2)$. F belongs to the domain of attraction of a Gaussian law (α=2) if and only if $E[X^2 I(|X|\leq x)]$ is a slowly varying function $L(x)$. F belongs to the normal domain of attraction of a stable law, denoted by $F \in D_N(S(\alpha,\beta,s))$, if and only if $F \in D(S(\alpha,\beta,s))$ with $L(x) = c$ for some constant $c > 0$. These are precisely the distribution functions whose sample means (properly normalized) have limiting distributions, i.e., $F \in D(S(\alpha,\beta,s))$ (respectively $D_N(S(\alpha,\beta,s))$) if and only if there exist $A_n > 0$ and $C_n \in \mathbb{R}$ such that $A_n^{-1}(n\bar{X}_n - C_n)$ converges in law, in which case the limiting distribution $\mathcal{L}_\infty$ is $S(\alpha,\beta,s)$ and $A_n = n^{1/\alpha}L(n)$ (respectively $A_n = n^{1/\alpha}$).

When $F \in D(S(\alpha,\beta,s))$, it can be easily verified that the statements (A.2) and (A.3) (ii) in Theorem A can be derived from Theorem 1. Namely there exist appropriate scaling a_n and resample size m_n such that the bootstrapped sample mean has asymptotic limiting distribution as given in Theorem 1, depending on the resample size. On the other hand, if the bootstrapped sample mean has a limiting distribution, then the limit, the scaling and the resample size necessarily must satisfy certain conditions described as follows.

Theorem 3. Assume $F \in D(S(\alpha,\beta,s))$ has infinite variance and assumptions (1.0)-(1.2), (1.3) hold. Then $\{a_n\}$ satisfies

$$\lim_{n\to\infty} m_n a_n^{-\alpha} L(a_n) = \text{constant} \qquad \text{if } m_n/n \longrightarrow c \in [0,\infty),$$

$$\lim_{n\to\infty} n(a_n\sqrt{n/m_n})^{-\alpha}L(a_n\sqrt{n/m_n}) = \text{constant} \qquad \text{if } m_n/n \longrightarrow \infty.$$

Moreover, if $\alpha=2$, then H is a nonrandom normal distribution.

Now we examine the performance of the averaged bootstrap with Athreya's (1987a) data-driven scaling.

Theorem 4. If $F \in D(S(\alpha,\beta,s))$, $\alpha \neq 2$, $m_n = n$, and Athreya's data-driven scaling $a_n(\mathcal{X}_n) = X_{n,n}$ (the sample maximum) is used, then (1.3) holds and G(x) has characteristic function

$$\varphi_G(t) = \frac{c_1 \exp(f_t(1))}{c_1 + \int_{-\infty}^{1}[1 - \exp(f_t(x))]d\lambda_\alpha(x)},$$

where $f_t(x) = e^{itx}-1-ita(x)$, $a(x) = \tau(x)$ if $\alpha \neq 1$, $a(x) = \tau(x)I(|x|\leq 1)$ if $\alpha = 1$, and the measure λ_α satisfies for all $x > 0$,

$$\lambda_\alpha[x,\infty) = c_1x^{-\alpha}, \quad \lambda_\alpha(-\infty,-x] = c_2x^{-\alpha}.$$

Since φ_G is quite different from the characteristic function of $\mathcal{L}_\infty$, the bootstrap fails even on the average with this scaling. Note that this data-based scaling results in a bootstrap algorithm which uses no specific features of the population distribution F other than the knowledge that it belongs to the domain of attraction of some stable law. However with an appropriate nonrandom scaling, the averaged-bootstrap algorithm can be improved in the sense that G is stable with the same index as $\mathcal{L}_\infty$ but with generally different skewness and scale.

Theorem 5. If $F \in D(S(\alpha,\beta,s))$, $1 \neq \alpha \in (0,2)$, $m_n = n$ and a_n is a nonrandom scaling based on the population distribution as follows:

$$n\, P(X \geq a_n) \xrightarrow[n\to\infty]{} 1,$$

then (1.3) holds and G(x) is $S(\alpha,\beta d_2(\alpha),sd_1(\alpha))$, where

$$d_1(\alpha) = \begin{cases} e^{-1}\left[\sum_{k=1}^{\infty} \frac{k^\alpha}{(k+1)!} + 1\right], & 1<\alpha<2, \\ e^{-1} \sum_{k=1}^{\infty} \frac{k^\alpha}{k!}, & 0<\alpha<1, \end{cases}$$

$$d_2(\alpha) = \begin{cases} 1 - \frac{2}{ed_1(\alpha)}, & 1<\alpha<2, \\ 1, & 0<\alpha<1. \end{cases}$$

When $\alpha=1$, G(x) is not Cauchy even with the above nonrandom scaling, and its characteristic function has a form more complex than that in Theorem 3 and is therefore not shown here.

Note that in each of the intervals (0,1) and (1,2), $d_1(\alpha)$ is increasing, infinitely differentiable with all derivatives positive and $d_1(0+) = 1-e^{-1}$, $d_1(1-) = 1$, $d_1(1+) = 2/e$, $d_1(2-) = 1$; see Figure 1 (c=1). Therefore the scale of G is smaller than that of $\mathcal{L}_\infty$. The factor $d_2(\alpha)$ is also increasing on (1,2) with $d_2(1+)=0$, $d_2(2-) = 1-2/e$; see Figure 2 (c=1). Hence the skewness $\beta d_2(\alpha)$ of G is the same as the skewness β of $\mathcal{L}_\infty$ for $\alpha \in (0,1)$, and has the same sign but is reduced by a factor of at least .736 for $\alpha \in (1,2)$.

When F belongs to the normal domain of attraction of a stable law, the bootstrap algorithm can in some cases be appropriately modified so that the averaged bootstrap will work. The simplest cases are when $0<\alpha<1$ (so there is no reduction in skewness) or when the skewness is known, e.g., in the symmetric case. Then only a scale adjustment to the algorithm, depending on the index α, is necessary; when α is unknown, an appropriate estimate may be used.

<u>Theorem 6</u>. If $F \in D_N(S(\alpha,\beta,s))$, $m_n=n$, either $0 < \alpha < 1$ or $\beta = 0$ and $1 \neq \alpha \in (0,2)$, and $a_n(\mathcal{X}_n) = [n\, d_1(\hat\alpha_n(\mathcal{X}_n))]^{1/\hat\alpha_n(\mathcal{X}_n)}$, where $\hat\alpha_n(\mathcal{X}_n)$ is an estimate

of α satisfying $(\hat{\alpha}_n-\alpha)\log n \xrightarrow[n\to\infty]{P} 0$, then (1.3) holds and $G = \mathcal{L}_\infty$, both being $S(\alpha,\beta,s)$.

For instance the estimate of α given in Hall (1982) works in Theorem 6.

Notice the skewness of G in Theorem 5, $\beta_G = \beta d_2(\alpha)$, is always smaller in absolute value than the skewness β of $\mathcal{L}_\infty$ when $1<\alpha<2$. Hence if prior to bootstrapping we transform the original data into new data belonging to the domain of attraction of a stable law with skewness $\beta d_2^{-1}(\alpha)$, and if we then appropriately adjust the standardization as in Theorem 6, using estimates of α and β if necessary, we obtain a bootstrap algorithm which works on the average provided $|\beta| \le d_2(\alpha)$.

<u>Theorem 7</u>. Assume $F \in D_N(S(\alpha,\beta,s))$, $1 \ne \alpha \in (0,2)$, $|\beta| \le d_2(\alpha)$, and $\hat{\alpha}_n, \hat{\beta}_n$ are consistent estimates of α and β, satisfying

$$(\hat{\alpha}_n-\alpha)\log n \xrightarrow[n\to\infty]{P} 0 \quad \text{if } 0 < \alpha < 1,$$

$$n^{1-1/\alpha}(\hat{\alpha}_n-\alpha) \xrightarrow[n\to\infty]{P} 0,\; n^{1-1/\alpha}(\hat{\beta}_n-\beta) \xrightarrow[n\to\infty]{P} 0 \quad \text{if } 1 < \alpha < 2.$$

Transform the data $\mathcal{X}_n$ to $\mathcal{Z}_n$ as follows:

$$Z_i = b^+(\hat{\alpha}_n,\hat{\beta}_n)\, X_i^+ - b^-(\hat{\alpha}_n,\hat{\beta}_n)\, X_i^- ,$$

where

$$b^+ = b^+(\alpha,\beta) = (1-\beta)[1+\beta/d_2(\alpha)],\quad b^- = b^-(\alpha,\beta) = (1+\beta)[1-\beta/d_2(\alpha)] ,$$

and consider the bootstrapped sample mean $T_n^* = a_n^{-1}\sum_{i=1}^n (Y_i - c_n)$, where

$$a_n = [n d_1(\hat{\alpha}_n)(1-\hat{\beta}_n^2)]^{1/\hat{\alpha}_n},\qquad c_n = \begin{cases} \overline{Z}_n & \text{if } \alpha > 1, \\ 0 & \text{if } \alpha < 1, \end{cases}$$

$m_n = n$, and $\{Y_i\}_{i=1}^n$ are resampled from $\{Z_i\}_{i=1}^n$. Then (1.3) holds and $G = \mathcal{L}_\infty$, both being $S(\alpha,\beta,s)$.

For instance the estimates of α and β given by Zolotarev (1986) work here.

Up to now we have discussed in detail that when $F \in D(S(\alpha,\beta,s))$ and $m_n = n$, the expected value G of the limiting distribution H is α-stable, with possibly different skewness and scale from $\mathcal{L}_\infty$, and how to make bootstrap resampling work a.s. by using the averaged-bootstrap algorithm. The following theorem shows, without any restriction on the resample size m_n, that G is α-stable if and only if F belongs to the domain of (partial) attraction of an α-stable law.

Theorem 8. Assume (1.0)-(1.2) and (1.3) hold.

(i) If $m_n/n \longrightarrow c \in (0,\infty]$, then $G(x)$ is α-stable with $1 \neq \alpha \in (0,2]$ if and only if $F \in D(S(\alpha,\beta,s))$.

(ii) If $m_n/n \longrightarrow 0$, then $G(x)$ is α-stable with $1 \neq \alpha \in (0,2]$ if and only if F belongs to the domain of $\{m_n\}$-partial attraction of an α-stable law ($F \in \{m_n\}\text{-}D_p(S(\alpha,\beta,s))$, i.e., $A_n^{-1}(m_n\overline{X}_{m_n} - C_n)$ converges in law for some $A_n > 0$ and $C_n \in \mathbb{R}$).

In either case $G(x)$ is Gaussian with mean zero and variance $d(2,c)$ if $\alpha = 2$, otherwise

$$G(x) \quad \text{is} \quad \begin{cases} S(\alpha,\ \beta,\ sd(\alpha,0)) & \text{if} \quad m_n/n \longrightarrow 0, \\ S(\alpha,\ \beta d_2(\alpha,c),\ sd(\alpha,c)d_1(\alpha,c)) & \text{if} \quad m_n/n \longrightarrow c \in (0,\infty), \\ S(\alpha,\ 0,\ sd(\alpha,\infty)[2^{\alpha/2}\cos(\alpha\pi/4)]^{-1}) & \text{if} \quad m_n/n \longrightarrow \infty, \end{cases}$$

where

$$d(\alpha,c) = \begin{cases} \lim\limits_{n\to\infty} m_n a_n^{-\alpha} L(a_n) & \text{if } m_n/n \longrightarrow c \in [0,\infty), \\ \lim\limits_{n\to\infty} n(a_n\sqrt{n/m_n})^{-\alpha} L(a_n\sqrt{n/m_n}) & \text{if } m_n/n \longrightarrow \infty, \end{cases}$$

$$d_1(\alpha,c) = \begin{cases} e^{-c} \sum_{k=0}^{\infty} \frac{c^k}{k!} [(k+1-c)^{\langle\alpha-1\rangle} - (k-c)^{\langle\alpha-1\rangle}], & 1<\alpha<2, \\ e^{-c} c^{-1} \sum_{k=1}^{\infty} \frac{c^k}{k!} k^{\alpha}, & 0<\alpha<1, \end{cases}$$

$$d_2(\alpha,c) = \begin{cases} \frac{e^{-c}}{d_1(\alpha,c)} \sum_{\substack{k=0 \\ k\neq c-1,c}}^{\infty} \frac{c^k}{k!} [|k+1-c|^{\alpha-1} - |k-c|^{\alpha-1}], & 1<\alpha<2, \\ 1 , & 0<\alpha<1, \end{cases}$$

and $x^{\langle p\rangle}=|x|^p\mathrm{sgn}(x)$.

Therefore if the resample size m_n satisfies $m_n/n \longrightarrow c \in [0,\infty)$, the averaged-bootstrap algorithm works with the appropriate standardization given in Theorem 3, using appropriate adjustments on the skewness and scale parameters as in Theorems 6 and 7, provided $F \in D_N\ (S(\alpha,\beta,s))$. When $m_n/n \longrightarrow \infty$, the averaged-bootstrap distribution eventually loses all skewness and becomes symmetric. In this case the averaged bootstrap works when $F \in D_N(S(\alpha,0,s))$, again using appropriate standardization and adjustment on the scale parameter as in Theorem 6.

Figures 1 through 5 illustrate the behavior of the distortion factors d_1 and d_2, as functions of $\alpha \in (0,1)\cup(1,2)$ and of $c > 0$. For fixed α, there is essentially no distortion (i.e., $d_1 \approx 1$ and $d_2 \approx 1$) when the resampling proportion c is sufficiently small (although the rate of convergence may be very slow, e.g., Figure 3); this agrees intuitively with the case $m_n/n \longrightarrow 0$. Distortion in the scale parameter can also be nearly eliminated by taking $m_n/n \approx 2/3$, regardless of the value of $\alpha \in (1,2)$ (see Figures 1 and 3). For $\alpha\in(1,2)$, the distortions in both scale and skewness increase dramatically as c initially increases from zero; for $c>1$, these distortions remain fairly stable (see Figures 3 and 5). In general there is less distortion as we approach the

FIG. 1: $d_1(a,c)$ vs. a

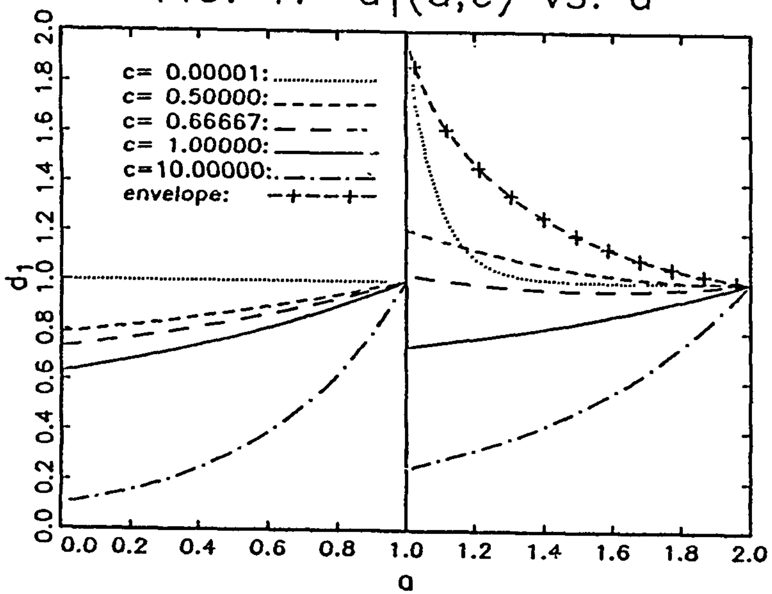


FIG. 2: $d_2(a,c)$ vs. a

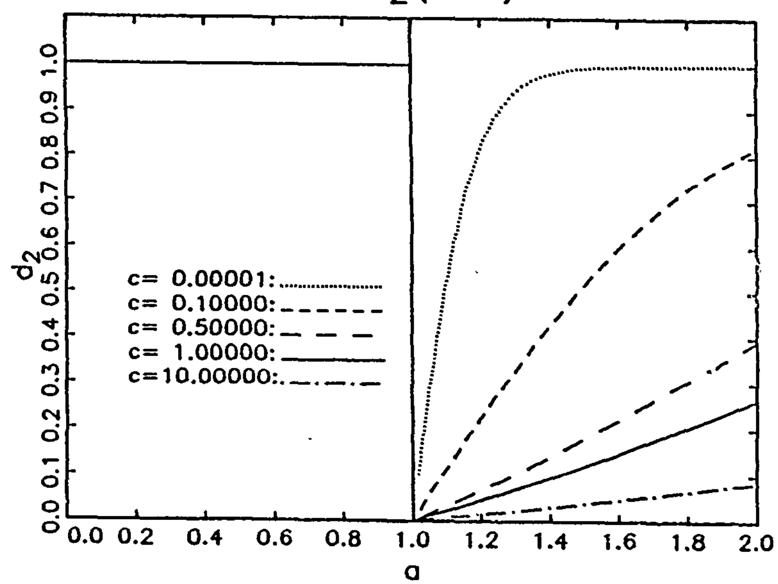


FIG. 3: $d_1(a,c)$ vs. c (a>1)

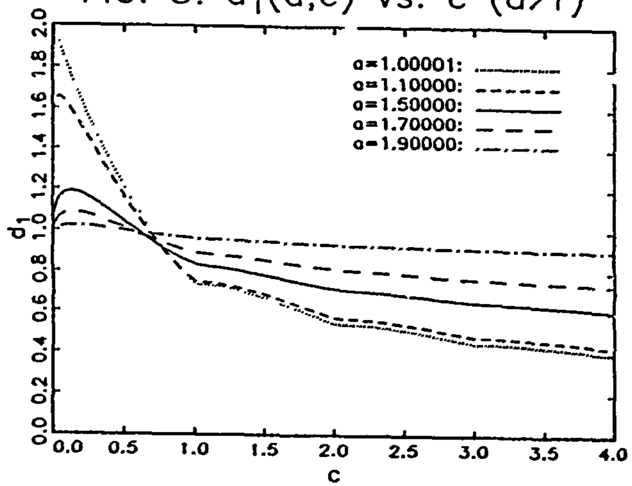


FIG. 4: $d_1(a,c)$ vs. c (a<1)

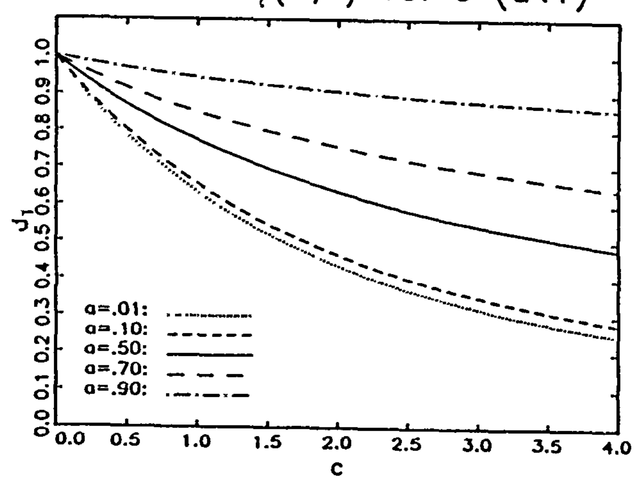
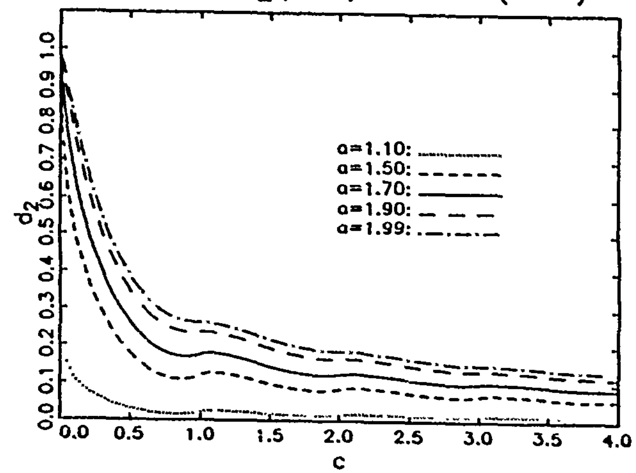


FIG. 5: $d_2(a,c)$ vs. c (a>1)



Gaussian case, i.e., $\alpha \uparrow 2$ (see Figures 1,2,3,5).

5. PROOFS

5.1. PROOF OF THEOREM 1.

We provide the proof only in the case where F has finite mean; when the mean of F does not exist or equals $\pm\infty$, the proof is essentially the same, in fact simpler since $c_n = 0$.

Athreya (1987a), Theorem A, shows that (1.3) is equivalent to

$$\{\varphi_{H_n}(t_j),\ j=1,\dots,k\} \xrightarrow[n\to\infty]{\mathcal{L}} \{\varphi_H(t_j),\ j=1,\dots,k\} \tag{5.1}$$

for all $k \geq 1$, $t_j \in \mathbb{R}$, $j=1,\dots,k$, where

$$\varphi_{H_n}(t) := E(e^{itT_n^*}|\mathcal{X}_n) = [1 + \frac{1}{m_n}\psi_n(t)]^{m_n},$$

with

$$\psi_n(t) := \frac{m_n}{n} \Sigma_{j=1}^n f_t\{a_n^{-1}(X_j-\overline{X}_n)\}$$

$$= (\int_{|x|\leq\epsilon} + \int_{|x|>\epsilon})f_t(x)dN_n(x) =: {}_{n,\epsilon}J(t)+J_{n,\epsilon}(t), \tag{5.2}$$

$f_t(x) = e^{itx}-1-itx$, for any Borel set $A \subset \mathbb{R}$,

$$N_n(A) := \frac{m_n}{n} \Sigma_{j=1}^n I_A\{a_n^{-1}(X_j-\overline{X}_n)\}, \tag{5.3}$$

and for any $x \in \mathbb{R}$, $N_n(x) := N_n([x,\infty))$ if $x \geq 0$, and $N_n(x) := N_n((-\infty,x])$ if $x < 0$. In turn (5.1) is equivalent to

$$\{\psi_n(t_j),\ j=1,\dots,k\} \xrightarrow[n\to\infty]{\mathcal{L}} \{\log \varphi_H(t_j),\ j=1,\dots,k\} \tag{5.4}$$

(see Theorem 5.5 in Billingsley (1968)), and by the Cramér-Wold device, to

$$\sum_{j=1}^{k} b_j \psi_n(t_j) \xrightarrow[n\to\infty]{\mathcal{L}} \sum_{j=1}^{k} b_j \log \varphi_H(t_j)$$

for arbitrary $b_1, \ldots, b_k$.

We find the necessary and sufficient conditions for k=1, i.e., for $\psi_n(t) \xrightarrow[n\to\infty]{\mathcal{L}} \log \varphi_H(t)$ for fixed $t \in \mathbb{R}$. The conditions and the proof are essentially the same for any k.

In a series of lemmas we find necessary and sufficient conditions for the convergence in law of all finite dimensional distributions (fdd's) of ${}_{n,\epsilon}J(t)$ and $J_{n,\epsilon}(t)$ in (5.2), and then prove that all fdd's of their sum converge in law if and only if those of both ${}_{n,\epsilon}J(t)$ and $J_{n,\epsilon}(t)$ converge in law.

We define ordered convergence in law, and in probability, of doubly indexed random variables $X_{n,\epsilon}$ as follows:

$$\mathcal{L} - \lim_{\epsilon\to 0} \lim_{n\to\infty} X_{n,\epsilon} = X \quad \text{iff} \quad \forall\, t \in \mathbb{R},\ \lim_{\epsilon\to 0} \lim_{n\to\infty} E\, e^{itX_{n,\epsilon}} = E\, e^{itX},$$

$$P - \lim_{\epsilon\to 0} \lim_{n\to\infty} X_{n,\epsilon} = X \quad \text{iff} \quad \forall\, \delta > 0,\ \lim_{\epsilon\to 0} \lim_{n\to\infty} P(|X_{n,\epsilon}-X| > \delta) = 0.$$

It can be easily verified that many classical results for r.v.'s with one index are applicable here for the ordered limit, such as: convergence in probability implies convergence in law, Slutsky's Theorem, and the Central Limit Theorem.

The first two lemmas provide the necessary and sufficient conditions for the convergence in law of the fdd's of ${}_{n,\epsilon}J(t)$ as $n \longrightarrow \infty$ and then $\epsilon \longrightarrow 0$.

Lemma 1. For each fixed $t \neq 0$,

$$P - \lim_{\epsilon\to 0} \lim_{n\to\infty} \frac{{}_{n,\epsilon}J(t)}{-\int_{|x|\le\epsilon} \frac{1}{2} t^2 x^2 dN_n(x)} = 1.$$

Hence all fdd's of ${}_{n,\epsilon}J(t)$ converge in law as $n \longrightarrow \infty$ and then $\epsilon \longrightarrow 0$ iff

$$(5.5) \qquad \int_{|x|\le\epsilon} x^2 dN_n(x) = \frac{m_n}{n} \sum_{j=1}^{n} a_n^{-2} (X_j-\bar{X}_n)^2 \, I(a_n^{-1}|X_j-\bar{X}_n| \le \epsilon)$$

converges in law, as $n \longrightarrow \infty$ and then $\epsilon \longrightarrow 0$, and the former limit equals $-t^2/2$ times the latter.

Since $\bar{X}_n$ converges a.s. to the finite mean of F, and $m_n a_n^{-2} \underset{n\to\infty}{\longrightarrow} 0$ by (1.0), standard arguments enable us to delete $\bar{X}_n$ in (5.5). Therefore all fdd's of ${}_{n,\epsilon}J(t)$ converge in law if and only if $\bar{X}_{n,\epsilon} := \sum_{j=1}^{n} X^j_{n,\epsilon}$ converges in law, where

$$X^j_{n,\epsilon} := \frac{m_n}{n} a_n^{-2} X_j^2 \, I(a_n^{-1}|X_j| \le \epsilon).$$

Note that $\{X^j_{n,\epsilon}\}_{1\le j\le n}$ form an infinitesimal array of independent r.v.'s, so we can apply Central Limit Theorem 4.7 in Araujo and Giné (1980) to obtain the following characterization.

Lemma 2. $\bar{X}_{n,\epsilon}$ converges in law as $n \longrightarrow \infty$ and then $\epsilon \longrightarrow 0$ if and only if conditions (1.b) and (1.c) are satisfied. In this case the characteristic function of the limit Z^2 is

$$\varphi_{Z^2}(u) = \exp\{iu\sigma^2 + \int_0^\infty (e^{iux}-1)d\nu(x)\}.$$

Corollary. All fdd's of ${}_{n,\epsilon}J(t)$ converge in law if and only if (1.b) and (1.c) are satisfied, and the limit equals $-\frac{1}{2}t^2Z^2$.

Note from (1.b) that $\nu \equiv 0$ when m_n/n is bounded, because the event $\{\, |X| \le \epsilon a_n,\ |X| > a_n \sqrt{\delta n/m_n} \,\}$ eventually becomes null for any fixed positive δ, when ϵ is sufficiently small. In this case $Z^2 = \sigma^2$ a.s., while in general $Z^2-\sigma^2$ is infinitely divisible with Lévy measure ν.

We now find necessary and sufficient conditions for the convergence in law of all fdd's of $J_{n,\epsilon}(t)$. It follows from (5.2) that the sample paths of

$\exp\{\psi_n(t)\}$ (or $\exp\{J_{n,\epsilon}(t)\}$) are characteristic functions of infinitely divisible laws with Lévy measures the corresponding sample paths of $N_n(x)$ (or $N_n(x)I(|x|>\epsilon)$). We show in Lemma 3 that the convergence in law of all fdd's of $J_{n,\epsilon}(t)$ is equivalent to the convergence in law of all fdd's of $N_n(x)$. Hence one can focus attention on $N_n(x)$, which is much simpler than $\psi_n(t)$ and essentially equivalent to a sum of independent random variables.

Lemma 3. Under Assumptions (1.0) - (1.2), the following are equivalent.

(i) For all $k \geq 1$ and $t_1,\ldots,t_k \in \mathbb{R}$,

$$\mathcal{L}\text{-}\lim_{\epsilon\to 0}\lim_{n\to\infty} \{J_{n,\epsilon}(t_j),\ j=1,\ldots,k\} = \{J(t_j),\ j = 1,\ldots,k\}.$$

(ii) For all $k \geq 1$ and $x_1,\ldots,x_k \in \mathbb{R}\backslash\{0\}$,

$$\{N_n(x_j),\ j=1,\ldots,k\} \xrightarrow[n\to\infty]{\mathcal{L}} \{N(x_j),\ j=1,\ldots,k\}.$$

(iii) Condition (1.a) in Theorem 1 holds.

Under any of these conditions, the limit in (i) is given by

$$J(t) = \begin{cases} \int_{-\infty}^{\infty} f_t(x)d\lambda(x) & \text{if } m_n/n \longrightarrow 0, \\ c\int_{-\infty}^{\infty} f_t(x)dN(x) & \text{if } m_n/n \longrightarrow c \in (0,\infty), \\ 0 & \text{otherwise}, \end{cases}$$

where N is a Poisson random measure with intensity measure $c^{-1}\lambda$.

Notice from Lemmas 2 and 3 that at least one of ${}_{n,\epsilon}J(t)$ or $J_{n,\epsilon}(t)$ will have a nonrandom limit, for all different choices of resample size m_n. Therefore the sum $\psi_n(t)$ will converge in law when both of them converge.

Lemma 4: Under Assumptions (1.0)-(1.2), (5.4) is equivalent to the convergence in law of all fdd's of ${}_{n,\epsilon}J(t)$ and $J_{n,\epsilon}(t)$, and if (5.4) holds then

$$\log \varphi_H(t) \stackrel{\mathcal{L}}{=} \lim_{\epsilon\to 0}\lim_{n\to\infty} {}_{n,\epsilon}J(t) + \lim_{\epsilon\to 0}\lim_{n\to\infty} J_{n,\epsilon}(t).$$

Proof of Lemma 1. Fix $\delta > 0$. Then, for each n and $\epsilon < 3\delta/|t|$ we have

$$P\left[\left| \frac{{}_{n,\epsilon}J(t)}{-\int_{|x|\le\epsilon} \frac{1}{2}t^2x^2 dN_n(x)} - 1 \right| > \delta\right]$$

$$= P[\ |\int_{|x|\le\epsilon} (e^{itx}-1-itx + \tfrac{1}{2}t^2x^2)dN_n(x)| > \delta \int_{|x|\le\epsilon} \tfrac{1}{2}t^2x^2 dN_n(x)\ ]$$

$$\le P[\ \int_{|x|\le\epsilon} \tfrac{1}{6}|tx|^3 dN_n(x) > \delta \int_{|x|\le\epsilon} \tfrac{1}{2}t^2x^2 dN_n(x)\ ]$$

$$\le P[\ \tfrac{1}{2}t^2(\tfrac{1}{3}|t|\epsilon-\delta) \int_{|x|\le\epsilon} x^2 dN_n(x) > 0\ ] = 0,$$

because $|e^{ix} - 1 - ix + x^2/2\ | \le |x|^3/6$ for all $x \in \mathbb{R}$. □

Proof of Lemma 2. We first verify that $\{X^j_{n,\epsilon},\ 1\le j\le n\}$ form an infinitesimal array. Indeed for fixed ϵ and $\delta > 0$, we have by (1.0),

$$P(X^1_{n,\epsilon} > \delta) = P(a_n^{-1}|X| \le \epsilon,\ \frac{m_n}{n} a_n^{-2}X^2 > \delta) \le P(|X| > a_n\sqrt{\delta n/m_n}) \xrightarrow[n\to\infty]{} 0.$$

Now by Theorem 4.7 in Araujo and Giné (1980), $\Sigma^n_{j=1}X^j_{n,\epsilon}$ converges in law (as $n \longrightarrow \infty$ and then $\epsilon \longrightarrow 0$) if and only if the following three conditions are satisfied:

(i) there exists a Lévy measure ν such that for all $0 \le \delta < u$ with $\nu\{\delta\} = 0 = \nu\{u\}$,

$$\lim_{\epsilon\to 0}\lim_{n\to\infty} n\ P(\delta < X^1_{n,\epsilon} \le u) = \nu(\delta,u];$$

(ii) for all $\delta > 0$ such that $\nu\{\delta\} = 0$, there exists $\sigma^2_\delta \ge 0$ such that

$$\lim_{\epsilon\to 0}\lim_{n\to\infty} n\ E[X^1_{n,\epsilon}\ I(X^1_{n,\epsilon} < \delta)] = \sigma^2_\delta;$$

(iii) there exists $a \in \mathbb{R}$ such that

$$\lim_{\delta\to 0} \overline{\lim_{\epsilon\to 0}}\ \overline{\lim_{n\to\infty}}\ n \operatorname{Var}[X^1_{n,\epsilon}\ I(X^1_{n,\epsilon} < \delta)] = a^2;$$

in this case the characteristic function of the limit is

$$\exp\{iu\sigma^2 - \tfrac{1}{2}u^2a^2 + \textstyle\int_0^\infty (e^{iux}-1)d\nu(x)\},$$

where $\sigma^2 = \lim_{\delta\downarrow 0} \sigma^2_\delta$, which exists since σ^2_δ is monotonic (decreasing) as $\delta \downarrow 0$.

Conditions (i) and (ii) are seen to be the same as (1.b) and (1.c) in Theorem 1, and (iii) is implied by (ii) since

$$n\, E[X^1_{n,\epsilon} I(X^1_{n,\epsilon} < \delta)]^2 \leq \delta n\, E[X^1_{n,\epsilon} I(X^1_{n,\epsilon} < \delta)] \underset{\substack{n\to\infty \\ \epsilon\to 0}}{\longrightarrow} \delta\sigma^2_\delta \underset{\delta\downarrow 0}{\longrightarrow} 0 .$$

Hence $a^2 = 0$, i.e., the Gaussian component in the limit is degenerate, and the limit has ch.f. as shown. □

Proof of Lemma 3. (ii) is equivalent to (iii): We will show that (iii) is equivalent to

(5.6) $\quad (N_n[y_1,\infty),\ N_n(-\infty,-y_2])$ converges in law for all $y_1,y_2 > 0$,

since the multivariate argument can be similarly carried out to achieve (ii).

Again by a standard argument, we can drop $\overline{X}_n$ in N_n, so for any $y_1,y_2 > 0$, $u_1,u_2 \in \mathbb{R}$,

(5.7) $\quad E\exp\{i(u_1N_n[y_1,\infty)+u_2N_n(-\infty,-y_2])\}$

$$\cong E\exp\{i\,\frac{m_n}{n}\ \Sigma^n_{j=1}[u_1I(a_n^{-1}X_j \geq y_1) + u_2\ I(a_n^{-1}X_j \leq -y_2)]\}$$

$$= [\ E\exp\{i\,\frac{m_n}{n}\,[u_1I(a_n^{-1}X \geq y_1) + u_2\ I(a_n^{-1}X \leq -y_2)]\}\ ]^n$$

$$= [1 + \frac{n}{n}\{\ [\exp(iu_1m_n/n) -1]P(a_n^{-1}X\geq y_1) + [\exp(iu_2m_n/n) -1]P(a_n^{-1}X\leq -y_2)\}\ ]^n.$$

Hence (5.6) is equivalent to

$$n[\exp(ium_n/n) -1]P(a_n^{-1}X \geq y) \quad \text{and} \quad n[\exp(ium_n/n) -1]P(a_n^{-1}X < -y)$$

converging for all $y > 0$, $u \in \mathbb{R}$, which is equivalent to (1.a). Therefore (5.6) is equivalent to (iii). Moreover (5.7) converges to

$$\begin{cases} \exp\{i[u_1\lambda[y_1,\infty) + u_2\lambda(-\infty,-y_2]]\} & \text{if } m_n/n \longrightarrow 0, \\ \exp\{c^{-1}[\exp(iu_1c)-1]\lambda[y_1,\infty) + c^{-1}[\exp(iu_2c)-1]\lambda(-\infty,-y_2]\} & \text{if } m_n/n \longrightarrow c\in(0,\infty), \\ 1 & \text{otherwise.} \end{cases}$$

<u>(i) implies (ii)</u>: For simplicity of notation we prove (ii) is implied by the convergence in law of all fdd's of $\psi_n(t)$.

From (5.2) we can write, with $g_t(u) = e^{itu}-1$,

$$\psi_n(t) = \int_{-\infty}^{\infty} f_t(x)dN_n(x)$$

$$= \int_0^{\infty} [it\int_0^x g_t(u)du]\ dN_n(x) + \int_{-\infty}^{0}[-it\int_x^0 g_t(u)du]\ dN_n(x)$$

$$= it\int_0^{\infty} g_t(x)N_n[x,\infty)dx - it\int_{-\infty}^{0} g_t(x)N_n(-\infty,x]dx$$

$$= -t^2\int_{-\infty}^{\infty} e^{itx}M_n(x)dx =: -t^2\psi_n^*(t), \tag{5.8}$$

where

$$M_n(x) = \begin{cases} \int_x^{\infty} N_n[u,\infty)du, & x > 0, \\ \int_{-\infty}^{x} N_n(-\infty,u]du, & x < 0. \end{cases} \tag{5.9}$$

Thus $\psi_n^*(t)$ is the Fourier transform of $M_n(x)$, which is piecewise linear, continuous with compact support.

It is clear from (5.2) that $\exp\{\psi_n(t)\}$ is a (random) characteristic function of an infinitely divisible law with (random) Lévy measure N_n, and (i) implies the fdd's of $\psi_n(t)$ converge in law to the fdd's of $\log \varphi_H(t) =: \psi(t)$. If the ψ_n's were nonrandom or more generally if the convergence were a.s.

instead of in law, the convergence of their Lévy measures in (ii) would follow immediately from classical results.

In order to establish (ii) in the current case, we apply the embedding theorem of Skorohod (1956) to the ψ_n's and ψ, whose sample functions are in $C(\mathbb{R})$. In this special case of processes with continuous paths, this implies that there exist stochastic processes $\{\tilde{\psi}_n(t)\}_{n\geq 1}$ and $\tilde{\psi}(t)$ on some probability space $(\tilde{\Omega},\tilde{\mathcal{F}},\tilde{P})$, such that: the fdd's of $\tilde{\psi}$ and ψ are the same, and so are those of $\tilde{\psi}_n$ and ψ_n for each n, and the sample paths of $\tilde{\psi}_n(t)$ and $\tilde{\psi}(t)$ are with probability one in $C(\mathbb{R})$ (therefore their corresponding induced measures on $C(\mathbb{R})$ are the same), and

$$(5.10) \qquad \tilde{\psi}_n(t) \xrightarrow[n\to\infty]{a.s.} \tilde{\psi}(t) \quad \text{for all } t.$$

Therefore there exists $\tilde{\Omega}_1 \subset \tilde{\Omega}$ with $\tilde{P}(\tilde{\Omega}_1) = 0$, such that for all $\tilde{\omega} \in \tilde{\Omega}\backslash\tilde{\Omega}_1$, (5.10) holds.

We will show that each $\tilde{\psi}_n$ has the same form as ψ_n, so that $\exp\{\tilde{\psi}_n\}$ is the characteristic function of an infinitely divisible law with Lévy measure $\tilde{N}_n$ having the same fdd's as N_n. Then (5.10) will imply a.s. convergence of $\tilde{N}_n$, therefore the convergence in law of all fdd's of N_n.

Since ψ_n and $\tilde{\psi}_n$ have the same induced measures on $C(\mathbb{R})$, in order to establish that $\tilde{\psi}_n$ has the same form as ψ_n, determined by (5.8) and (5.9), it suffices to show that the inverse of the map that defines ψ_n or ψ_n^* from N_n via M_n is measurable. We now proceed to discuss this.

We first formalize the two maps from N_n to M_n, and then from M_n to ψ_n^*. Let $\mathcal{M}$ (or $\mathcal{N}$) be the set of all discrete signed measures (or discrete measures) on $\mathbb{R}$ with finitely many atoms with the topology of weak convergence. Let $\mathcal{F}$ be the set of all functions that are piecewise linear with finitely many segments, continuous, with compact support. $\mathcal{F}$ is a measurable subset of $C(\mathbb{R})$ with the uniform topology. Define $T_1: \mathcal{M} \longrightarrow \mathcal{F}$ by

$$(T_1\nu)(x) = \begin{cases} \int_x^\infty \nu[y,\infty)dy, & x \geq 0, \\ \int_{-\infty}^x \nu(-\infty,y]dy, & x < 0, \end{cases}$$

for all $\nu \in \mathcal{M}$. T_1 is one-to-one and onto and its inverse $T_1^{-1}: \mathcal{F} \longrightarrow \mathcal{M}$ is defined by $(T_1^{-1}f)(a,b] = (Bf)(b)-(Bf)(a)$, where $B, B_\eta: \mathcal{F} \longrightarrow K(\mathbb{R})$ are given by

$$(Bf)(x) = \lim_{\eta \downarrow 0} (B_\eta f)(x), \qquad (B_\eta f)(x) = \frac{1}{\eta}[f(x+\eta)-f(x)].$$

B is a measurable map since B_η is continuous for each fixed $\eta > 0$. Therefore T_1^{-1} is well defined and measurable.

The relationship (5.8) is the Fourier transform $T_2: C_0(\mathbb{R}) \longrightarrow C(\mathbb{R})$, where $C(\mathbb{R})$ (or $C_0(\mathbb{R})$) is the set of all continuous functions on $\mathbb{R}$ (with compact support). $T_2\{C_0(\mathbb{R})\} = T_2\{\cup_{m=1}^\infty C_0[-m,m]\} = \cup_{m=1}^\infty T_2\{C_0[-m,m]\}$ is a Borel subset of $C(\mathbb{R})$, since the sets $C_0[-m,m]$ of all continuous functions with support on $[-m,m]$ are closed subsets of $C(\mathbb{R})$, and T_2 is continuous on each $C_0[-m,m]$ as

$$\|T_2(f_1)-T_2(f_2)\|_\infty \leq \int |f_1(x)-f_2(x)|dx \leq 2m\, \|f_1-f_2\|_\infty.$$

Hence we can define the inverse map $T_2^{-1}: T_2\{C_0(\mathbb{R})\} \longrightarrow C_0(\mathbb{R})$ by

$$(T_2^{-1}\hat{f})(x) = \lim_{\sigma\to 0} \frac{1}{2\pi} \int_{-\infty}^\infty e^{-itx}\, \hat{f}(t)\exp\{-\tfrac{1}{2}\sigma^2 t^2\}dt =: \lim_{\sigma\to 0} (A_\sigma \hat{f})(x),$$

for each $\hat{f} \in T_2\{C_0(\mathbb{R})\}$ (see e.g., Rudin (1966), § 9.7). T_2^{-1} is Borel measurable since each map A_σ is continuous on $T_2\{C_0(\mathbb{R})\}$:

$$\|A_\sigma(\hat{f}_1) - A_\sigma(\hat{f}_2)\|_\infty \leq |\hat{f}_1-\hat{f}_2\|_\infty \frac{1}{2\pi} \int_{-\infty}^\infty \exp\{-\tfrac{1}{2}\sigma^2 t^2\}dt = \frac{1}{2\pi\sigma} \|\hat{f}_1-\hat{f}_2\|_\infty .$$

Therefore we have

$$\mathcal{N} \subset \mathcal{M} \underset{T_1^{-1}}{\overset{T_1}{\underset{\longleftarrow}{\overset{\longrightarrow}{(1\text{-}1)}}}} \mathcal{F} \subset C_0(\mathbb{R}) \underset{T_2^{-1}}{\overset{T_2}{\underset{\longleftarrow}{\overset{\longrightarrow}{(1\text{-}1)}}}} T_2\{C_0(\mathbb{R})\} \subset C(\mathbb{R}).$$

Then $B = T_2\{T_1(\mathcal{N})\} = (T_2^{-1})^{-1}\{(T_1^{-1})^{-1}(\mathcal{N})\}$ is a measurable subset of $C(\mathbb{R})$. Since the paths of all ψ_n^*'s are in B with probability one by (5.8) and (5.9), we have

$$1 = P \circ \psi_n^{*-1}(B) = \tilde{P} \circ \tilde{\psi}_n^{*-1}(B),$$

which implies $\tilde{\psi}_n^*(\cdot,\tilde{\omega})$ belongs to B a.s. $(\tilde{P})$. Thus there exists $\tilde{\Omega}_0 \subset \tilde{\Omega}$ with $\tilde{P}(\tilde{\Omega}_0) = 0$, such that for all $\tilde{\omega} \in \tilde{\Omega}\backslash\tilde{\Omega}_0$, we have

$$\tilde{N}_n(\cdot,\tilde{\omega}) := T_1^{-1}\{T_2^{-1}[\tilde{\psi}_n^*(\cdot,\omega)]\} \in \mathcal{N}, \quad n = 1,2,\ldots$$

i.e.,

$$\tilde{\psi}_n(t) = \int f_t(x)d\tilde{N}_n(x), \quad n=1,2,\ldots$$

Therefore for all $\tilde{\omega} \in \tilde{\Omega}\backslash(\tilde{\Omega}_1 \cup \tilde{\Omega}_0)$ and all t, we have

$$\exp\{\int f_t d\tilde{N}_n\} = \exp\{\tilde{\psi}_n(t)\} \underset{n\to\infty}{\longrightarrow} \exp\{\tilde{\psi}(t)\}. \tag{5.11}$$

Applying the classical result, the Lévy measures $\tilde{N}_n$ of $\exp\{\tilde{\psi}_n(t)\}$ must converge a.s. $(\tilde{P})$ to the Lévy measure $\tilde{N}$ of $\exp\{\tilde{\psi}(t)\}$.

Now notice that N_n and $\tilde{N}_n$ have the same fdd's because for all Borel subsets A of $K(\mathbb{R})$

$$\tilde{P}(\tilde{N}_n \in A) = \tilde{P}(\tilde{\psi}_n^* \in (T_2^{-1})^{-1}\{(T_1^{-1})^{-1}(A)\}) = P(\psi_n^* \in (T_2^{-1})^{-1}\{(T_1^{-1})^{-1}(A)\}) = P(N_n \in A).$$

Therefore for all $x_1,\ldots,x_k \in \mathbb{R}\backslash\{0\}$,

$$(N_n(x_1),\ldots,N_n(x_k)) \overset{\mathcal{L}}{=} (\tilde{N}_n(x_1),\ldots,\tilde{N}_n(x_k)) \xrightarrow[n\to\infty]{\text{a.s. } (\tilde{P})} (\tilde{N}(x_1),\ldots,\tilde{N}(x_k))$$

which implies (iii) by the previous argument. Hence $\tilde{N} \overset{\mathcal{L}}{=} N$ for some Poisson random measure N with intensity measure λ. Since $\tilde{N}$ is a random Lévy measure,

$\int_{-1}^{1} x^2 d\tilde{N}(x) < \infty$ a.s. $(\tilde{P})$ and thus $\int_{-1}^{1} x^2 dN(x) < \infty$ a.s. (P). Therefore the form of its ch.f. implies that $\int_{-1}^{1} x^2 d\lambda(x) < \infty$, and hence λ is a Lévy measure.

(ii) implies (i): We write

$$J_{n,\epsilon}(t) = \int_{|x|\geq M} f_t(x)dN_n(x) + \int_{M>|x|\geq\epsilon} f_t(x)dN_n(x) =: J_{n,M}(t) + J_{n,\epsilon,M}(t).$$

For the first term we have

$$|J_{n,M}(t)| \leq 2|t|\int_{|x|\geq M} |x|dN_n(x)$$

$$= 2|t| \sum_{j=1}^{n} \frac{m_n}{n} a_n^{-1}|X_j-\bar{X}_n| \; I(a_n^{-1}|X_j-\bar{X}_n| \geq M)$$

$$\tilde{=} 2|t| \sum_{j=1}^{n} \frac{m_n}{n} a_n^{-1}|X_j| \; I(a_n^{-1}|X_j| \geq M) =: 2|t| \sum_{j=1}^{n} W_{n,M}^{j} \; ,$$

because by a standard argument we can drop the sample mean. $\{W_{n,M}^{j}\}_{j=1}^{n}$ is an infinitesimal array as $n \to \infty$ and then $M \to \infty$, since for any $\delta > 0$,

$$P(W_{n,M}^{1} > \delta) = P(|X| \geq a_n M, \; |X| \geq \delta a_n n/m_n) \leq P(|X| \geq a_n M) \xrightarrow[n\to\infty]{} 0.$$

Therefore, applying the degenerate version of the C.L.T. (Loève (1977), p. 329), we have

$$(*) \qquad P - \lim_{M\to\infty} \lim_{n\to\infty} \sum_{j=1}^{n} W_{n,M}^{j} = 0$$

if and only if the following three conditions are satisfied:

(a) $\lim_{M\to\infty} \lim_{n\to\infty} n \, E[W_{n,M}^{1} \, I(W_{n,M}^{1} \leq \delta)] = 0$ for some $\delta > 0$,

(b) $\lim_{M\to\infty} \lim_{n\to\infty} n \, P(W_{n,M}^{1} > \delta) = 0$ for all $\delta > 0$,

(c) $\lim_{M\to\infty} \lim_{n\to\infty} n \, \mathrm{Var}[W_{n,M}^{1} \, I(W_{n,M}^{1} \leq \delta)] = 0$ for some $\delta > 0$.

It can easily be verified that (a) and (b) are equivalent to assumptions (1.1)-(1.2), and (c) is implied by (a) due to:

$$\mathrm{Var}[W^1_{n,M}\ I(W^1_{n,M} \le \delta)] \le E[W^1_{n,M}\ I(W^1_{n,M} \le \delta)]^2 \le \delta\ E[W^1_{n,M}\ I(W^1_{n,M} \le \delta)].$$

It follows that (*) holds. Hence the fdd's of $J_{n,\epsilon}(t)$ converge in law (as $n\to\infty$ and then $\epsilon \to 0$) if and only if those of $J_{n,\epsilon,M}(t)$ do (as $n\to\infty$, then $\epsilon \to 0$, and then $M \to \infty$), and their limits are the same.

Now $f_t(x)$ is continuous and bounded on $[-M,M]$ for each fixed $M > 0$, hence the convergence in law of all fdd's of N_n implies that of $J_{n,\epsilon,M}$ as $n \to \infty$, $\epsilon \to 0$, and then $M \to \infty$:

$$J_{n,\epsilon,M}(t) \xrightarrow[n\to\infty]{\mathcal{L}} \int_{\epsilon\le|x|<M} f_t dN \xrightarrow[\epsilon\to 0]{\mathcal{L}} \int_{|x|<M} f_t dN \xrightarrow[M\to\infty]{\mathcal{L}} J(t). \qquad \square$$

<u>Proof of Lemma 4</u>. (5.4) implies the convergence in law of all fdd's of N, and hence of $J_{n,\epsilon}$ (Lemma 3). Notice

$$\psi_n(t) \overset{\mathcal{L}}{=} \tilde{\psi}_n(t) = \left(\int_{|x|\le\epsilon} + \int_{|x|>\epsilon}\right) f_t d\tilde{N}_n =: {}_{n,\epsilon}\tilde{J}(t) + \tilde{J}_{n,\epsilon}(t),$$

where the fdd's of $\tilde{\psi}_n(t)$ converge a.s. $(\tilde{P})$ to those of $\tilde{\psi}(t)$ (see Lemma 3). Since $\tilde{N}_n$ converges a.s. $(\tilde{P})$ to $\tilde{N}$, the Lévy measure of $\exp\{\tilde{\psi}(t)\}$, $\tilde{J}_{n,\epsilon}(t)$ will converge a.s. $(\tilde{P})$ as $n \longrightarrow \infty$ and then $\epsilon \longrightarrow 0$ because $\tilde{N}_n(x)I(|x| > \epsilon)$ is again a Lévy measure for fixed $\epsilon > 0$, and

$$\lim_{\epsilon\to 0}\lim_{n\to\infty} \tilde{J}_{n,\epsilon}(t) = \lim_{\epsilon\to 0}\int_{|x|>\epsilon} f_t d\tilde{N} = \int f_t d\tilde{N} \quad \text{a.s. } (\tilde{P}).$$

Therefore ${}_{n,\epsilon}\tilde{J}(t) = \tilde{\psi}_n(t) - \tilde{J}_{n,\epsilon}(t)$ converges a.s. $(\tilde{P})$ as $n \longrightarrow \infty$ and then $\epsilon \longrightarrow 0$ to $\tilde{\psi}(t) - \int f_t d\tilde{N}$. (Note $\tilde{\psi}(t)$ may have a random Gaussian component $-\frac{1}{2}t^2\sigma^2(w)$ which distinguishes $\tilde{\psi}(t)$ from $\int f_t d\tilde{N}$). Furthermore,

$${}_{n,\epsilon}J(t) = \int_{|x|\le\epsilon} f_t dN_n \overset{\mathcal{L}}{=} \int_{|x|\le\epsilon} f_t d\tilde{N}_n = {}_{n,\epsilon}\tilde{J}(t)$$

since N_n and $\tilde{N}_n$ have the same fdd's for all $x \in \mathbb{R}\backslash\{0\}$, and $f_t(0) = 0$. Therefore the fdd's of ${}_{n,\epsilon}J(t)$ must converge in law.

Conversely, if the fdd's of ${}_{n,\epsilon}J(t)$ and $J_{n,\epsilon}(t)$ converge in law, then so do those of their sum ${}_{n,\epsilon}J(t) + J_{n,\epsilon}(t)$, since: In each of the two mutually exclusive cases (i) $m_n/n \longrightarrow c \in [0,\infty)$, and (ii) otherwise, either ${}_{n,\epsilon}J(t)$ or $J_{n,\epsilon}(t)$ converges in law to a nonrandom limit, and therefore converges in probability, thus ensuring the convergence in law of the sum by Slutsky's Theorem. □

5.2. PROOFS OF THEOREMS 2-8.

<u>Proof of Theorem 2</u>. Since for each fixed $x \in \mathbb{R}$, ${}_1H_{k_n}(x) \xrightarrow[n\to\infty]{\mathcal{L}} H(x)$, and $0 \leq {}_1H_{k_n}, H \leq 1$, it follows that $E\overline{H}_n(x) = E_1H_{k_n}(x) \xrightarrow[n\to\infty]{} G(x)$. On the other hand, by the inequality of Hoeffding (1963), for all $\epsilon > 0$,

$$P(|\overline{H}_n(x) - E\overline{H}_n(x)| > \epsilon) = P\left(\left| \frac{1}{\ell_n} \sum_{j=1}^{\ell_n} [{}_jH_{k_n}(x) - E_jH_{k_n}(x)]\right| > \epsilon\right)$$

$$\leq 2\exp\{-2\ell_n\epsilon^2\} \leq 1/n^2,$$

for sufficiently large n. The Borel-Cantelli Lemma implies a.s. convergence of $\overline{H}_n$, and uniform convergence follows in the standard way. □

<u>Proof of Theorem 3</u>. Put $\chi(t) = 1-F(t)+F(-t)$ for $t > 0$.

(i) $\alpha = 2$. We have (Feller (1966), p. 545)

$$\lim_{t\to\infty} \frac{t^2\chi(t)}{E[X^2I(|X| \leq t)]} = \frac{2-\alpha}{\alpha} = 0. \tag{5.12}$$

<u>Case (a)</u>: $m_n/n \longrightarrow c \in [0,\infty)$. (1.a)-(1.c) imply

$$m_n\chi(a_ny) \xrightarrow[n\to\infty]{} \lambda[y,\infty) + \lambda(-\infty,-y],$$

$$m_n a_n^{-2} E[X^2 \ I(|X| \le \epsilon a_n)] = m_n a_n^{-2} L(a_n \epsilon) \xrightarrow[\epsilon\to 0]{n\to\infty} \sigma^2 .$$

These imply $\lambda \equiv 0$ by (5.12), and $m_n a_n^{-2} L(a_n) \longrightarrow \sigma^2$ (≥ 0).

Case (b): $m_n/n \longrightarrow \infty$. Put $b_n = a_n\sqrt{n/m_n}$. Then $b_n \longrightarrow \infty$ by (1.0). Also (1.a)-(1.c) imply for all $\delta > 0$, $n\chi(b_n\sqrt{\delta}\,) \longrightarrow \nu(\delta,\infty)$, and

$$\sigma_\delta^2 = \lim_{n\to\infty} m_n a_n^{-2} EX^2 I(|X| \le b_n\sqrt{\delta}\,) = \lim_{n\to\infty} m_n a_n^{-2} L(b_n\sqrt{\delta}) = \lim_{n\to\infty} nb_n^{-2} L(b_n\sqrt{\delta}) = \lim_{n\to\infty} nb_n^{-2} L(b_n).$$

Again using (5.12), we have $\nu \equiv 0$, and $nb_n^{-2} L(b_n) \xrightarrow[n\to\infty]{} \sigma^2 \ge 0$. In either case, H is Gaussian.

(ii) $0 < \alpha < 2$. Case (a): $m_n/n \longrightarrow c \in [0,\infty)$. (1.a) implies $m_n(ya_n)^{-\alpha} L(a_n y)$ converges for all $y > 0$; thus $m_n a_n^{-\alpha} L(a_n) \longrightarrow$ constant (≥ 0).

Case (b): $m_n/n \longrightarrow \infty$. (1.b) implies $n\,\chi(b_n\sqrt{\delta}) \approx (c_1+c_2)n(b_n\sqrt{\delta})^{-\alpha} L(b_n\sqrt{\delta})$ converges for all $\delta > 0$. □

Proof of Theorem 4. By Athreya (1987a),

$$\varphi_G(t) = E\exp\{\textstyle\int_{-\infty}^{\infty} f_t(x)dN(x)\},$$

where $N(A) = N'(A\tau)$, N' is a Poisson r.m. with intensity λ_α, and τ is the last jump of N'. Hence

$$\varphi_G(t) = \exp\{f_t(1)\}\ E\exp\{\textstyle\int_{(-\infty,1)} f_t(x)dN(x)\} =: \exp\{f_t(1)\}\ \tilde{\varphi}(t).$$

It can be shown that, given τ, N is a Poisson r.m. with intensity measure $\lambda_\alpha(x\tau) = \tau^{-\alpha}\lambda_\alpha(x)$, and that τ has max-stable distribution for which

$$E\exp\{u\tau^{-\alpha}\} = c_1\alpha \textstyle\int_0^\infty \exp\{(u-c_1)y^{-\alpha}\}\ y^{-\alpha-1}dy = [1-c_1^{-1}u]^{-1}$$

if $\mathrm{Re}(u) < c_1$. Hence

$$\tilde{\varphi}(t) = E\ [\ E\ \{\exp\{\int_{(-\infty,1)} f_t(x)dN(x)\}\ |\ \tau\ \}]$$

$$= E\exp\{\tau^{-\alpha}\int_{-\infty}^{1} [\exp\{f_t(x)\}-1]\, d\lambda_\alpha(x)\}$$

$$= [\ 1 - c_1^{-1}\int_{-\infty}^{1} [\exp\{f_t(x)\}-1]\, d\lambda_\alpha(x)\ ]^{-1},$$

since the real part of the last integrand is $\leq 0 < c_1$. □

<u>Proof of Theorem 5</u>. In this case, Athreya (1986a) showed

$$\varphi_H(t) = \exp\{\int_{-\infty}^{\infty} f_t(x)dN(x)\},$$

where N is a Poisson r.m. with intensity λ_α. Therefore

$$\varphi_G(t) = \exp\{\ \int_{-\infty}^{\infty} [\exp\{f_t(x)\}-1]d\lambda_\alpha(x)\ \}.$$

When $1 < \alpha < 2$, we find

$$-\log\varphi_G(t) = \alpha[c_1\int_0^{\infty} + c_2\int_{-\infty}^{0}](1 - \exp\{f_t(x)\})\ \frac{dx}{|x|^{1+\alpha}}$$

$$= \alpha|t|^\alpha\ [\ c_1\int_0^{\infty} \{1 - \exp[\cos u - 1 + i(\sin u - u)\mathrm{sgn}(t)]\ \}\frac{du}{u^{1+\alpha}}$$

$$+ c_2\int_0^{\infty} \{1 - \exp[\cos u - 1 - i(\sin u - u)\mathrm{sgn}(t)]\}\frac{du}{u^{1+\alpha}}\ ]$$

$$= |t|^\alpha\ a(c_1+c_2)[1 - i\ \frac{b}{a}\ \frac{c_1-c_2}{c_1+c_2}\ \mathrm{sgn}(t)],$$

where

$$a = \alpha\int_0^{\infty} [1 - \exp(\cos u - 1)\cos(\sin u - u)]\ \frac{du}{u^{1+\alpha}},$$

$$b = \alpha\int_0^{\infty} \exp(\cos u - 1)\sin(\sin u - u)\ \frac{du}{u^{1+\alpha}}.$$

Integrating by parts and then using the series expansion of $\exp\{e^{iu}\}$, we obtain

$$a - ib = \frac{\Gamma(2-\alpha)}{(\alpha-1)e}\ \{\ -2\cos\frac{\alpha\pi}{2} - \exp(i\frac{\alpha\pi}{2})[\sum_{k=1}^{\infty} \frac{k^\alpha}{(k+1)!} - 1]\ \},$$

hence $b/a = tg(\alpha\pi/2)d_2(\alpha)$. Since $a(c_1+c_2) = sd_1(\alpha)$, $\beta = (c_1-c_2)/(c_1+c_2)$, $s = (c_1+c_2)\Gamma(2-\alpha)|\cos(\alpha\pi/2)|/(\alpha-1)$, it follows that

$$\varphi_G(t) = \exp\{ -sd_1(\alpha)|t|^\alpha[1 - i\beta d_2(\alpha)tg(\alpha\pi/2)sgn(t)] \}.$$

The proof for $0 < \alpha < 1$ is similar with $a - ib = e^{-i\alpha\pi/2}\Gamma(1-\alpha)d_1(\alpha)$. □

Proof of Theorem 6. If $a_n = [n\, d_1(\hat{\alpha}_n)]^{1/\hat{\alpha}_n}$, then we have

$$a_n^{-1}\, n^{1/\alpha} = \exp\{ (\alpha\hat{\alpha}_n)^{-1}(\hat{\alpha}_n-\alpha)\log n \}\, d_1^{-1/\hat{\alpha}_n}(\hat{\alpha}_n) \xrightarrow[n\to\infty]{P} d_1^{-1/\alpha}(\alpha),$$

for $d_1^{-1/\alpha}(\alpha)$ is continuous over $(0,1)$. Therefore, for $A = [x,\infty)$, we have

$$P[N_n(A) = k] = P[a_n^{-1}(X_{n,n-k+1}-\bar{X}_n) \geq x, \quad a_n^{-1}(X_{n,n-k} - \bar{X}_n) < x]$$

$$= P[a_n^{-1}\, n^{1/\alpha}\, n^{-1/\alpha}(X_{n,n-k+1} - \bar{X}_n) \geq x,\; a_n^{-1}\, n^{1/\alpha}\, n^{-1/\alpha}(X_{n,n-k} - \bar{X}_n) < x]$$

$$\xrightarrow[n\to\infty]{} P(N(A) = k),$$

by Slutsky's Theorem, where N is a Poisson r.m. with intensity $\lambda_\alpha d_1^{-1}(\alpha)$. Therefore it can be shown (see Proposition 1, Athreya (1987a)) that

$$P - \lim_{\epsilon\to 0}\lim_{n\to\infty}\, {}_{n,\epsilon}J(t) = 0, \qquad P - \lim_{M\to\infty}\lim_{n\to\infty} J_{n,M}(t) = 0.$$

$$\psi_n(t) \xrightarrow[n\to\infty]{\mathcal{L}} \int f_t(x)dN(x),$$

and similarly for their fdd's.

Applying the proof of Theorem 5, we find

$$\varphi_G(t) = \exp\{ -sd_1^{-1}(\alpha)d_1(\alpha)|t|^\alpha[1 - i\beta tg(\alpha\pi/2)sgn(t)] \}. \qquad □$$

Proof of Theorem 7. Since $X = X^+ - X^-$ is in $D(S(\alpha,\beta,s))$, it follows that

$Z = b^+X^+ - b^-X^-$ is in $D(S(\alpha,\beta_Z,s_Z))$, where

$$\beta_Z = (c_1b^+ - c_2b^-)/(c_1b^+ + c_2b^-) = \beta/d_2,$$

$$s_Z = s(c_1b^+ + c_2b^-)/(c_1+c_2) = s(1-\beta^2).$$

Note that

$$\varphi_{H_n}(t) = \{\ 1 + \frac{1}{n}\ \Sigma_{j=1}^n\ f_t[a_n^{-1}(Z_j - \overline{Z}_n)]\ \}^n$$

converges in law if and only if

$$\Sigma_{j=1}^n\ f_t[a_n^{-1}(Z_j - \overline{Z}_n)] \tag{5.14}$$

converges in law. Notice $\{Z_j\}_{j=1}^n$ are no longer iid due to the estimates $\hat{\alpha}_n$ and $\hat{\beta}_n$, which depend on the entire sample $\mathfrak{X}_n$. Hence we must show that (5.14) asymptotically behaves as when α and β are known; then we can handle the problem similarly as in Theorem 5. The arguments are as follows.

Let $Z_{0i} = b^+X_i^+ - b^-X_i^-$, $\overline{Z}_{0n} = b^+\overline{X_n^+} - b^-\overline{X_n^-}$. Then using $|f_t(x) - f_t(y)| \le 2|t||x-y|$, and writing $\hat{b}_n^{\pm}$ for $b^{\pm}(\hat{\alpha}_n,\hat{\beta}_n)$, we have

$$\begin{aligned}
&|\ \Sigma_{j=1}^n\ f_t[a_n^{-1}(Z_j - \overline{Z}_n)] - \Sigma_{j=1}^n\ f_t[a_n^{-1}(Z_{0j} - \overline{Z}_{0n})]\ | \\
&\le 2|t|\ \Sigma_{j=1}^n a_n^{-1}|Z_j - \overline{Z}_n - Z_{0j} + \overline{Z}_{0n}| \\
&\le\ 2|t|a_n^{-1}\{|\hat{b}_n^+ - b^+|\ \Sigma_{j=1}^n |X_j^+ - \overline{X_n^+}| + |b^- - \hat{b}_n^-|\ \Sigma_{j=1}^n\ |X_j^- - \overline{X_n^-}|\} \\
&\le [a_n^{-1}\ n^{1/\alpha}]\ n^{-1/\alpha}\{|\hat{b}_n^+ - b^+|\ \Sigma_{j=1}^n X_j^+ + |\hat{b}_n^- - b^-|\Sigma_{j=1}^n X_j^-\}\ 4|t|\ . \qquad (5.15)
\end{aligned}$$

The first factor converges in probability to $[d_1(\alpha)(1-\beta^2)]^{-1/\alpha}$ using a similar argument as in Theorem 6.

When $0<\alpha<1$, the terms $|\hat{b}_n^+ - b^+|$, $|\hat{b}_n^- - b^-|$ converge in probability to 0 because $\hat{\alpha}_n$, $\hat{\beta}_n$ are consistent and $b^{\pm}(\alpha,\beta)$ are continuous; and the terms

$n^{-1/\alpha}\Sigma_{j=1}^{n}X_j^+$, $n^{-1/\alpha}\Sigma_{j=1}^{n}X_j^-$ converge in law since X^+, X^- also belong to a domain of attraction (with the same index α).

When $1<\alpha<2$, the terms $n^{-1}\Sigma_{j=1}^{n}X_j^+$, $n^{-1}\Sigma_{j=1}^{n}X_j^-$ converge a.s. by the SLLN; and the terms $n^{1-1/\alpha}|\hat{b}_n^+-b^+|$, $n^{1-1/\alpha}|\hat{b}_n^--b^-|$ converge to zero in probability since

$$\hat{b}_n^+-b^+ = \frac{1}{d_2(\hat{\alpha}_n)}\{ [1-\beta-\hat{\beta}_n-d_2(\hat{\alpha}_n)](\hat{\beta}_n-\beta) - \beta(1-\beta)[d_2(\hat{\alpha}_n)-d_2(\alpha)]/d_2(\alpha) \},$$

$$\hat{b}_n^--b^- = \frac{1}{d_2(\hat{\alpha}_n)}\{ -[1+\beta+\hat{\beta}_n-d_2(\hat{\alpha}_n)](\hat{\beta}_n-\beta) + \beta(1+\beta)[d_2(\hat{\alpha}_n)-d_2(\alpha)]/d_2(\alpha) \},$$

$\hat{\alpha}_n, \hat{\beta}_n$ are consistent estimates of α, β with rate $n^{1-1/\alpha}$, and $d_2(\cdot)$ is differentiable on $(1,2)$.

Therefore (5.15) converges to zero in probability, and (5.14) converges in law if and only if $\Sigma_{j=1}^{n} f_t[a_n^{-1}(Z_{0j}-\bar{Z}_{0n})]$ converges, and they have the same limits. We can now use arguments similar to those in Theorems 5 and 6 on $\{Z_{0j}\}_{j=1}^{n}$ to establish the result. □

<u>Proof of Theorem 8</u>. <u>Suffiency of both (i) and (ii)</u>: Assume $F \in D(S(\alpha,\beta,s))$, or $D_p(S(\alpha,\beta,s))$ when $m_n/n \longrightarrow 0$.

<u>Case (1)</u>: $m_n/n \longrightarrow c \in [0,\infty)$. Then by Theorem 3, (1.3) implies that $\lim_{n\to\infty} m_n a_n^{-\alpha}L(a_n) = d$ for some $d \geq 0$, and $G = H$, which is normal with $\sigma^2=d(2,c)$ when $\alpha = 2$; while when $1 \neq \alpha \in (0,2)$,

$$\sigma^2 = \lim_{\epsilon\to 0}\lim_{n\to\infty} m_n a_n^{-2} E[X^2 I(|X| \leq a_n\epsilon)]$$

$$= \lim_{\epsilon\to 0}\lim_{n\to\infty} m_n a_n^{-2} (a_n\epsilon)^{2-\alpha}L(a_n\epsilon) = \lim_{\epsilon\to 0} \epsilon^{2-\alpha} \lim_{n\to\infty} m_n a_n^{-\alpha}L(a_n) = 0.$$

Also by (1.a),

$$\lambda[y,\infty) = \lim_{n\to\infty} m_n P(X > ya_n) = \lim_{n\to\infty} c_1 m_n (ya_n)^{-\alpha}L(ya_n) = c_1 d y^{-\alpha},$$

and similarly $\lambda(-\infty,-y] = c_2 d y^{-\alpha}$, so that $\lambda = d\lambda_\alpha$. Hence by (1.5),

$$\varphi_G(t) = \begin{cases} \exp\{d\int_{-\infty}^{\infty} f_t(x)d\lambda_\alpha(x)\} & \text{if } m_n/n \longrightarrow 0, \\ \exp\{dc^{-1}\int_{-\infty}^{\infty} [\exp\{cf_t(x)\}-1]d\lambda_\alpha(x)\} & \text{if } m_n/n \longrightarrow c \in (0,\infty); \end{cases}$$

$$= \begin{cases} \exp\{-ds|t|^\alpha[1 - i\beta tg(\alpha\pi/2)sgn(t)]\} & \text{if } m_n/n \longrightarrow 0, \\ \exp\{-dsd_1(\alpha,c)|t|^\alpha[1 - i\beta d_2(\alpha,c)tg(\alpha\pi/2)sgn(t)]\} & \text{if } m_n/n \longrightarrow c \in (0,\infty); \end{cases}$$

using arguments similar to those in Theorem 5. In fact Theorem 5 is a special case of Theorem 8 with $c = 1$, $d = 1$, $d_1(\alpha) = d_1(\alpha,1)$, and $d_2(\alpha) = d_2(\alpha,1)$.

Case (2): $m_n/n \longrightarrow \infty$. Then by Theorem 3, $\lim_{n\to\infty} nb_n^{-\alpha}L(b_n) = d$ for some $d \geq 0$, where $b_n = a_n\sqrt{n/m_n}$. Moreover $G = H$ is normal with $\sigma^2 = d(2,c)$ if $\alpha = 2$, while when $1 \neq \alpha \in (0,2)$ we have from (1.b) (with $0<\delta<u$ such that $\nu\{\delta\} = 0 = \nu\{u\}$),

$$\nu(\delta,u] = \lim_{\epsilon\to 0}\lim_{n\to\infty} n\, P(b_n\sqrt{\delta} < |X| \leq b_n\sqrt{u}\,)$$

$$= \lim_{n\to\infty} (c_1+c_2)n[(b_n\sqrt{\delta})^{-\alpha}L(b_n\sqrt{\delta}) - (b_n\sqrt{u})^{-\alpha}L(b_n\sqrt{u})] = (c_1+c_2)d(\delta^{-\alpha/2}-u^{-\alpha/2}).$$

Condition (1.c) can be written as

$$\sigma_\delta^2 = \lim_{\epsilon\to 0}\lim_{n\to\infty} m_na_n^{-2}E\{X^2I(|X| \leq b_n\sqrt{\delta})\}$$

$$= \lim_{n\to\infty} nb_n^{-2}(b_n\sqrt{\delta})^{2-\alpha}L(b_n\sqrt{\delta}) = \delta^{1-\alpha/2}\lim_{n\to\infty} nb_n^{-\alpha}L(b_n) = d\delta^{1-\alpha/2} \xrightarrow[\delta\to 0]{} 0.$$

Therefore by (1.5), $\varphi_G(t) = E\exp\{-\tfrac{1}{2}t^2W\}$, where

$$\varphi_W(t) = \exp\{\, d(c_1+c_2)\tfrac{\alpha}{2}\int_0^\infty (e^{itx}-1)\,\frac{dx}{x^{1+\alpha/2}}\,\},$$

which is the ch.f. of $S(\alpha/2,1,ds)$. Hence it follows that

$$\varphi_G(t) = \exp\{-|t|^\alpha sd/[2^{\alpha/2}\cos(\alpha\pi/4)]\}.$$

Necessity of (ii): $m_n/n \longrightarrow 0$. If $\alpha = 2$, it follows from the expression of φ_G

in (1.5) that $\lambda \equiv 0$, i.e., for $y > 0$,

$$m_n P(a_n^{-1} X > y) \longrightarrow 0, \quad m_n P(a_n^{-1} X < -y) \longrightarrow 0.$$

Also there exists σ^2 such that

$$\sigma^2 = \lim_{\epsilon \to 0} \lim_{n \to \infty} m_n a_n^{-2} E\{X^2 I(|X| \leq \epsilon a_n)\}.$$

Hence $F \in D_p(S(2,0,s))$ (Ibragimov and Linnik (1971), Eq. 2.6.12-2.6.13, and Feller (1971), p. 555).

If $1 \neq \alpha \in (0,2)$, then $\sigma^2 = 0$, and λ is the Lévy measure of an α-stable law, which means there exist constants c_1, $c_2 \geq 0$, such that

$$\lambda[y,\infty) = c_1 y^{-\alpha}, \quad \lambda(-\infty,-y] = c_2 y^{-\alpha}.$$

Thus by condition (1.a),

$$\lim_{n \to \infty} m_n P(a_n^{-1} X > y) = c_1 y^{-\alpha}, \quad \lim_{n \to \infty} m_n P(a_n^{-1} X < -y) = c_2 y^{-\alpha},$$

and by (1.c),

$$\lim_{\epsilon \to 0} \lim_{n \to \infty} m_n a_n^{-2} E[X^2 I(|X| \leq \epsilon a_n)] = \sigma^2 = 0.$$

Again these imply $F \in D_p(S(\alpha,\cdot,\cdot))$ (see Ibragimov and Linnik (1971), Eqs. (2.6.3)-(2.6.5) and Feller (1971)).

<u>Necessity of (i)</u>: <u>Case (1)</u>: $m_n/n \longrightarrow c \in (0,\infty)$. The argument for $\alpha = 2$ is as in the necessity of (ii). Now if G is α-stable with $1 \neq \alpha \in (0,2)$, then by (1.5),

$$\psi(t) := \log \varphi_G(t) = -\frac{1}{2} t^2 \sigma^2 + c^{-1} \int_{-\infty}^{\infty} [\exp\{c f_t(x)\} - 1] d\lambda(x)$$

$$= -|t|^\alpha s\{1 - i\beta \operatorname{tg}(\alpha\pi/2) \operatorname{sgn}(t)\}.$$

First we show that $\sigma^2 = 0$. For $t > 0$, we have

$$(5.16) \quad \frac{1}{2} t^{2-\alpha} \sigma^2 + c^{-1} t^{-\alpha} \int_{-\infty}^{\infty} \{1 - \exp[c(\cos tx - 1)] \cos[c(\sin tx - tx)]\} d\lambda(x) = s.$$

Since $1-\exp[c(\cos tx-1)]\cos[c(\sin tx-tx)] \geq 0$ for all $t,x \in \mathbb{R}$, and $s \in (0,\infty)$, letting $t \longrightarrow \infty$ in (5.16) implies $\sigma^2 = 0$, for $t^{2-\alpha} \longrightarrow \infty$. Now $\exp\{\psi(t)\}$ is the ch.f. of an α-stable law, therefore it satisfies $n\psi(t) = \psi(n^{1/\alpha}t)$, i.e.,

$$n \int_{-\infty}^{\infty}[\exp\{cf_t(x)\} - 1]\, d\lambda(x) = \int_{-\infty}^{\infty}[\exp\{cf_t(x)\} - 1]\, d\lambda(n^{-1/\alpha}x),$$

where $\lambda(x) = \lambda[x,\infty)$ if $x > 0$, $\lambda(-\infty,x]$ if $x < 0$. Therefore $n\,\lambda(n^{1/\alpha}x) = \lambda(x)$. By the monotonicity of λ, we can conclude that $\lambda(y)$ is proportional to y^{ρ}, $-\infty<\rho<+\infty$, (Feller (1971), VIII 8, Lemma 3), and ρ has to be $-\alpha$. Therefore $F \in D(S(\alpha,\cdot,\cdot))$ by using similar arguments as in the necessity of (ii).

<u>Case (2)</u>: $m_n/n \longrightarrow \infty$. Since G is stable, and (1.5) implies that it must be symmetric, we have

$$E \exp\{-\tfrac{1}{2}\, t^2(\sigma^2+W)\} = \exp\{-s_0|t|^{\alpha}\},$$

which implies $\sigma^2 + W \sim S(\alpha/2,1,s_1)$, since the left hand side is the Laplace transform of $\sigma^2+ W$ at $t^2/2$. Therefore

$$E\exp\{it(\sigma^2 + W)\} = \exp\{it\sigma^2 + \int_0^{\infty} (e^{itx}-1)d\nu(x)\} = \exp\{-s_1|t|^{\alpha}\}.$$

Hence $\sigma^2 = 0$ and ν is the Lévy measure corresponding to $S(\alpha/2,1,s_1)$, i.e.,

$$\lim_{n\to\infty} n\, P(|X| > b_n\sqrt{\delta}) = c(\sqrt{\delta})^{-\alpha},$$

and with a similar argument as in Case (1),

$$0 = \sigma^2 = \lim_{\delta\to 0}\lim_{n\to\infty} nb_n^{-2}\, E[X^2\, I(|X| \leq b_n\sqrt{\delta})].$$

Hence $F_{|X|} \in D(S(\alpha,\cdot,\cdot))$, which implies $F \in D(S(\alpha,\cdot,\cdot))$ since

$$F \in D(S(\alpha,\cdot,\cdot)) \quad \text{iff} \quad \frac{t^2\chi(t)}{E[X^2I(|X|\leq t)]} \xrightarrow[t\to\infty]{} \frac{2-\alpha}{\alpha},$$

(see Ibragimov and Linnik (1971), Theorem 2.6.3). □

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